STOCHASTIC MODEL FOR A SEDIMENTATION TANK

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INTRODUCTION

The networks of sanitation carry mineral particles of such a size that it is possible to retain them owing to lined up sedimentation tanks.

The observations made by the users show that their efficiency is not optimum (8) (10). A precise analysis of the behaviours of such an apparatus is therefore necessary (9) (15) (11).

In our research, it is shown that it is roughly possible to schematize the history of sediments in two different phases (7) (2):
— on phase of settling linked to the mean velocity distribution;
— on phase of putting back into suspension of the deposited particles linked to the big whirling structures.

The study of this second phase offers a non negligible interest in so far as the frequency of the scour of such apparatus is very low. This phase must not be neglected in the estimation of their performance.

The analysis of the working is executed on a scale-model on which it is set obstacles, altering the structure of the flow. Some favour the initial sedimentation, others the non-putting back into suspension of this one.

We will use a technique of modelisation based on stochastic phenomena which seems to us to explain the physical phenomenon in the best manner (1) (4) (5) (14).

DESCRIPTION OF THE USED MEANS

Experimental set up (Fig. 1): It is composed of 2000 mm long, 400 mm large
and 300 mm deep sedimentation tank fed by a 4 m long pipe, the drainage of the waters is carried out by a 3 m long pipe having 1 cm in diameter. The axis of each pipe is 15 cm from the sedimentation tank floor. The supply is effected by a tank with a constant level placed at 12 m above the installation.

Hydraulic aspect:
— Three flows are studied: 5, 7 and 9 l/s; the level of the water is kept constant in the tank at 30 cm;
— Four kinds of obstacles are used, as drawed in Figure 2: without obstacles, inlet bars, inlet plate, inlet plate + bars;
— The 12 configurations are then tested and the comparison with each case is possible.

Sorts of used sediments: particles of Table 1 are put into the flow by the injector placed at the head of the installation and are recuperated by a system of interchangeable baskets, then dried and weighed.

Choice of used particles: the particles have been chosen to correspond to sands from 0.6 to 0.8 mm in diameter by a Froude similarity with λ from 3 to 5. The sedimentation tank corresponds then to a 6 m long prototype being able to clean flows from 80 to 150 l/s or to a 10 m long prototype and being able to clean flows from 300 to 500 l/s. The particles stay therefore on an average of 5 min to a day in the sedimentation tank according to studied the flow or obstacle.
Fig. 2. Types of configuration studied in the sedimentation tank

**TABLE 1**

**DESCRIPTION OF SOLID PARTICLES**

<table>
<thead>
<tr>
<th>Kind</th>
<th>Density $\text{kg/m}^3$</th>
<th>Shape</th>
<th>Diameter $\text{mm}$</th>
<th>Settling Velocity $\text{mm/s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blue</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Polystyrene</td>
<td>1035</td>
<td>Cylindrical</td>
<td>2</td>
<td>45</td>
</tr>
<tr>
<td>White</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Polystyrene</td>
<td>1025</td>
<td>Spherical</td>
<td>2</td>
<td>35</td>
</tr>
</tbody>
</table>

**ANALYTICAL MODELS**

*Introduction:* without trying to mark the geographical position of a particle in the sedimentation tank, we try to totally characterize the behaviour of
the solid phase referring to the concept of the transit time repartition function (T.T.R.).

Model of stochastic advance of solid particles in the sedimentation tank: the sedimentation tank is represented by the two dimensionnal lattice drawed on Figure 3.

The position of a particle at time \( t \) is given by \((X(t), Y(t))\). The nature of the phenomenon is a bivariate process for which we have \( \{X(0) = 0; Y(0) = J\}\).

![Diagram of lattice for sedimentation tank](image)

**Fig. 3. Lattice for the sedimentation tank**

The following assumptions are:
- all the particles have the same transport characteristics;
- the interacting effect between particles is negligeable.

With these assumptions, we have then a MARKOV process with two independant variables. So, in this paper, we shall confine our attention to the marginal process \( \{X(t); t \geq 0\} \) which also has the MARKOV property.

1) **Evaluation of** \( P_1(t) \) — The passing from one state to another is given by the diagram Figure 4.

- Injection is determined by state 0
- Outlet is determined by state \( K+1 \)
Fig. 4. Passing probability from one state to another

— Sedimentation is determined by state \(-1\)
— Particles in suspension having more or less advance are determined by states \(1, \ldots, k\).

— The probability that a particle advances is determined by: \(\lambda \Delta t\)
— The probability that a particle reverses is determined by: \(\mu \Delta t\)
— The probability that a particle sediments is determined by: \(\gamma \Delta t\)
— The probability that a particle changes directly from the sedimented state to the outlet state is determined by: \(\delta \Delta t\)

Let us note: \(P_i(t) = P\{X(t) = i\} \ i = -1, 0, 1, \ldots, k, k + 1\) probability that a particle is in \(i\) at time \(t\).

The calculation of \(P_i(t)\) is made by the following way; let us take that the passing passage probabilities have the following properties for \(\Delta t \to 0\):

\[
\begin{cases}
P\{X(t+\Delta t) = i+1/X(t) = i\} = \lambda \Delta t + 0(\Delta t) \\
P\{X(t+\Delta t) = i-1/X(t) = i\} = \mu \Delta t + 0(\Delta t) \\
P\{X(t+\Delta t) = -1/X(t) = i\} = \gamma \Delta t + 0(\Delta t)
\end{cases} \quad i = 1, \ldots, k
\]

Then: \(P\{X(t+\Delta t) = i/X(t) = i\} = 1-(\lambda + \mu + \gamma) \Delta t + 0(\Delta t)\) and, from this that:

\[
P\{X(t+\Delta t) = i\} = P\{X(t) = i-1\} \cdot P\{X(t+\Delta t) = i/X(t) = i-1\} + P\{X(t) = i\} \cdot P\{X(t+\Delta t) = i/X(t) = i\} + P\{X(t) = i+1\} \cdot P\{X(t+\Delta t) = i/X(t) = i+1\} + 0(\Delta t)
\]

from this after a simple algebraic transformation, we obtain \((\Delta t \to 0)\) for \(i = 1, \ldots, k\), which gives:
\[
\begin{bmatrix}
-1 & 0 & 1 & i & k-1 & k & k+1 \\
-\delta & \gamma & \gamma & \gamma & \gamma & \gamma & 0 \\
0 & -(\lambda+\gamma) & \mu & 0 & 0 & 0 & 0 \\
0 & \lambda & -(\lambda+\mu+\gamma) & \mu & 0 & 0 & 0 \\
0 & 0 & \lambda & -(\lambda+\mu+\gamma) & \mu & 0 & 0 \\
0 & 0 & 0 & \lambda & -(\alpha\lambda+\mu+\gamma) & 0 & 0 \\
0 & 0 & 0 & 0 & \alpha\lambda & 0 & 0
\end{bmatrix}
\begin{bmatrix}
P_{-1}(t) \\
P_0(t) \\
P_k(t) \\
P_{k-1}(t) \\
P_{k+1}(t)
\end{bmatrix}
\]

with: \( P_0(0) = 1; \ P_1(0) = 0 \ i \neq 0 \)

2) A particular case — Let us take \( \mu = 0 \). So, we obtain:

\[
P_{-1}(t) = \frac{\gamma}{\lambda + \gamma - \delta} e^{-\delta t} \sum_{i=0}^{k} \left( \frac{\lambda}{\lambda + \gamma - \delta} \right)^i \left[ 1 - \sum_{j=0}^{i} \frac{[(\lambda + \gamma - \delta)t]^j}{j!} e^{-(\lambda + \gamma - \delta)t} \right]
\]

\[
P_i(t) = \frac{(\lambda t)^i}{i!} e^{-(\lambda + \gamma)t} \quad i = 0, \ldots, k
\]

\[
P_{k+1}(t) = TTR = \frac{\gamma}{\lambda + \gamma - \delta} \sum_{i=0}^{k} \left( \frac{\lambda}{\lambda + \gamma - \delta} \right)^i (1 - e^{-\delta t}) + \left( \frac{\lambda}{\lambda + \gamma} \right)^{k+1}
\]

\[
\left[ 1 - \sum_{i=0}^{k} \frac{[(\lambda + \gamma)t]^i}{i!} e^{-(\lambda + \gamma)t} \right] - \frac{\gamma \delta}{(\lambda + \gamma)(\lambda + \gamma - \delta)} \sum_{i=0}^{k} \left( \frac{\lambda}{\lambda + \gamma - \delta} \right)^i
\]

\[
\sum_{j=0}^{i} \left( \frac{\lambda + \gamma - \delta}{\lambda + \gamma} \right)^j \left[ 1 - \sum_{n=0}^{i} \frac{[(\lambda + \gamma)t]^n}{n!} e^{-(\lambda + \gamma)t} \right]
\]

The curves of transit time repartition have the aspect shown on Figure 5.
The analysis of this kind of curve shows that 3 outlet phases of particle exist: a first one in which the particles only cross out the sedimentation tank, a second one in which the particles go out quickly and a third phase in which the particles go out slowly due to passing from state $-1$ to $k+1$, that we observe experimentally.

Let us notice that it is important to have $\delta \neq 0$ because, in all our experiments, the whole of the particles ends by going out. Having taking $\mu = 0$ is necessary to obtain a simple integration of the differential system, but it comes to neglect the recirculation stream. Finally the achieved answer is so complex, so difficult to work that it has led to introduce a simplified model.

**Simplified model:** let us take it in this part that only three states are possible for a solid particle:

— State 1: in suspension or in unstable zone of sedimentation
— State 2: outlet state
— State 3: in a zone of stable sedimentation

Then, let us consider the probabilities of passing from one state to the other (see Figure 6) where:

$-\lambda \Delta t$ is the probability that a particle gets out

$-\gamma \Delta t$ is the probability that a particle sediments

$-\delta \Delta t$ is the probability that a particle comes back into suspension.

In the same way as in the previous paragraph, we obtain the T.T.R.:

$$
\begin{pmatrix}
P_1(t) \\
P_2(t) \\
P_3(t)
\end{pmatrix} =
\begin{pmatrix}
-(\lambda + \gamma) & 0 & \delta \\
\lambda & 0 & 0 \\
\gamma & 0 & -\delta
\end{pmatrix}
\begin{pmatrix}
P_1(t) \\
P_2(t) \\
P_3(t)
\end{pmatrix}
$$
with the limit conditions $P_1(0) = 1; P_2(0) = 0; P_3(0) = 0$, that leads us to this solution:

$$
\begin{align*}
P_1(t) &= -\frac{\lambda+\gamma+\theta_2}{\theta_1-\theta_2} e^{\theta_1 t} + \frac{\lambda+\gamma+\theta_1}{\theta_1-\theta_2} e^{\theta_2 t} \\
P_2(t) &= \frac{\lambda+\gamma}{\theta_1-\theta_2} (1-e^{\theta_1 t}) - \frac{\lambda+\gamma}{\theta_1-\theta_2} \frac{1-e^{\theta_2 t}}{(\lambda+\gamma+\theta_1)(\lambda+\gamma+\theta_2)} = \text{T.T.R.} \\
P_3(t) &= -\frac{\lambda+\gamma}{\theta_1-\theta_2} \frac{1-e^{\theta_2 t}}{(\theta_1-\theta_2)} (e^{\theta_1 t} - e^{\theta_2 t})
\end{align*}
$$

with: $\theta_1 = \frac{-(\lambda+\gamma+\delta) - \sqrt{(\lambda+\gamma+\delta)^2 - 4 \delta \lambda}}{2}$

$\theta_2 = \frac{-(\lambda+\gamma+\delta) + \sqrt{(\lambda+\gamma+\delta)^2 - 4 \delta \lambda}}{2}$

and taking:

$$
\begin{align*}
\lambda_1 &= -\theta_1 \\
\lambda_2 &= -\theta_2 \\
\alpha &= \frac{\lambda}{\theta_1} \frac{\lambda+\gamma+\theta_2}{\theta_1-\theta_2} \\
\lambda &= \alpha \lambda_2 + (1-\alpha) \lambda_1 \\
\delta &= \frac{\lambda_1 \lambda_2}{\alpha \lambda_2 + (1-\alpha) \lambda_1} \\
\gamma &= \frac{(\alpha \lambda_2 + (1-\alpha)\lambda_1) (\alpha \lambda_1 + (1-\alpha)\lambda_2) - \lambda_1 \lambda_2}{\alpha \lambda_2 (1-\alpha) \lambda_1}
\end{align*}
$$

we have:

$$
\text{T.T.R.} = P_2(t) = 1 - \alpha e^{-\lambda_1 t} - (1 - \alpha) e^{-\lambda_2 t}
$$

Then it is noticed that the so simplified model expresses two phases of drainage of the particles:
## Table 2

**Obtained Value by Smoothing of Experimental Results**

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Without Obstacles</th>
<th>Bars</th>
<th>Inlet Baffle</th>
<th>Inlet Baffle + Bar</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \alpha )</td>
<td>( \lambda_1 )</td>
<td>( \beta )</td>
<td>( \beta_1 )</td>
</tr>
<tr>
<td>( 5 \text{ l/s} )</td>
<td>35</td>
<td>76</td>
<td>.15</td>
<td>17</td>
</tr>
<tr>
<td>White</td>
<td>27</td>
<td>.43</td>
<td>.43</td>
<td>9.5</td>
</tr>
<tr>
<td>( 5 \text{ l/s} )</td>
<td>28</td>
<td>48</td>
<td>.06</td>
<td>11</td>
</tr>
<tr>
<td>Blue</td>
<td>14</td>
<td>.21</td>
<td>.34</td>
<td>3.9</td>
</tr>
<tr>
<td>( 7 \text{ l/s} )</td>
<td>36</td>
<td>150</td>
<td>11</td>
<td>31</td>
</tr>
<tr>
<td>White</td>
<td>62</td>
<td>26</td>
<td>75</td>
<td>70</td>
</tr>
<tr>
<td>( 7 \text{ l/s} )</td>
<td>36</td>
<td>92</td>
<td>3.2</td>
<td>28</td>
</tr>
<tr>
<td>Blue</td>
<td>36</td>
<td>8.3</td>
<td>51</td>
<td>43</td>
</tr>
<tr>
<td>( 9 \text{ l/s} )</td>
<td>76</td>
<td>170</td>
<td>28</td>
<td>53</td>
</tr>
<tr>
<td>White</td>
<td>140</td>
<td>35</td>
<td>27</td>
<td>150</td>
</tr>
<tr>
<td>( 9 \text{ l/s} )</td>
<td>61</td>
<td>190</td>
<td>20</td>
<td>50</td>
</tr>
<tr>
<td>Blue</td>
<td>120</td>
<td>30</td>
<td>57</td>
<td>130</td>
</tr>
</tbody>
</table>
— a first phase corresponding to a primary decantation in the zones where the sediments stay rather stably. The proportion of the particles trapped like that is \((1 - \alpha)\); the other particles being drained off either without having never been deposited on staying deposited a short time. The proportion of particles being drained off is \(\alpha\) an average staying time of \(1/\lambda_1\);

— a second phase corresponding to a slow retaking of the sediments and to a drainage with a average staying time of \(1/\lambda_2\).

The three values \((\alpha, \lambda_1, \lambda_2)\) are obtained by the smoothing of the experimental results (see Table 2 and Figura 7).

**Tabla 3**

**VALUE OF THE PERFORMANCE RATE**

<table>
<thead>
<tr>
<th>F l o w</th>
<th>Kind of Particles</th>
<th>Without Obstacles</th>
<th>Bars</th>
<th>Inlet Baffle</th>
<th>Inlet Baffle + Bar</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 l/s</td>
<td>White</td>
<td>11</td>
<td>.9</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>5 l/s</td>
<td>Blue</td>
<td>4</td>
<td>.5</td>
<td>12</td>
<td>4</td>
</tr>
<tr>
<td>7 l/s</td>
<td>White</td>
<td>553</td>
<td>18</td>
<td>69</td>
<td>67</td>
</tr>
<tr>
<td>7 l/s</td>
<td>Blue</td>
<td>169</td>
<td>4</td>
<td>70</td>
<td>67</td>
</tr>
<tr>
<td>9 l/s</td>
<td>White</td>
<td>2070</td>
<td>1040</td>
<td>514</td>
<td>465</td>
</tr>
<tr>
<td>9 l/s</td>
<td>Blue</td>
<td>1150</td>
<td>468</td>
<td>444</td>
<td>294</td>
</tr>
</tbody>
</table>

**ANÁLISIS DE LOS RESULTADOS**

*First analysis:* we can first notice that the white particles always get out faster than the blue ones, and that \(\alpha\) for the white particles is always higher to \(\alpha\) for the blue ones; result that we could expect in so far as those ones have a lower settling velocity that these ones.

The adjunction of an obstacle always appreciably reduces the value of \(\alpha\) and, in this case, the inlet plate is more efficient in the improvement than the bars.

If we want the trapped particles to stay in the sedimentation tank, \(\lambda_2\)
must be as low as possible and, in this case, bars are more efficient in the improvement than inlet plates.

To achieve a compromise or even a general improvement on the two parameters we can simultaneously place two kinds of obstacles.

To appreciate the quality of a sedimentation tank to retain particles as long as possible (see Figure 7), we propose to use the parameter $r$ so defined.

**Performance rate:** experimentally, all the particles end by going out after enough time. In the usual meaning of efficiency (the ratio between the trapped particles and the whole of the particle inside the sedimentation tank) is equal to zero.

It is the ratio between the average staying time of the liquid particles and the average staying time of the solid particles.

\[ t_1 = \frac{V_o}{Q} \text{ with } V_o \text{ volume of the sedimentation tank and } Q \text{ the flow} \]

and 

\[ t_s = \int_0^\infty t (\alpha \lambda_1 e^{-\lambda_1 t} + (1-\alpha) \lambda_2 e^{-\lambda_2 t}) \, dt \]

hence: 

\[ r = \frac{V_o/Q}{\frac{\alpha}{\lambda_1} + \frac{1-\alpha}{\lambda_2}} \]

the lower the value of $r$, the better the working of the sedimentation tank.
Second analysis: so we obtain the value of \( r \) (see Table 3). This table shows that the adjunction of a plate is particularly interesting for heavy flows for which it is important to increase the initial trapping, but it is less interesting for the light flows for which this initial trapping is always rather and for which it is advisable to maintain the trapped particles, in that case, the bars are the most efficient.

Parameter \( r \) seems to be representative of the working and of the percentage of trapped materials after a working time of about 2000 s. This can be used for sewage systems since the sedimentation tank are scoured only periodically.

Comparison with Hazen (13) and Clements-Price (6) 's models

HAZEN's model allows us to calculate a limit for the settling velocity \( q_e \) of the particles which will decay at 100% (see Table 4).

<table>
<thead>
<tr>
<th>Table 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>LIMIT FOR THE SETTLING VELOCITY</td>
</tr>
<tr>
<td>WITH HAZEN'S MODEL</td>
</tr>
<tr>
<td>Flow</td>
</tr>
<tr>
<td>Settling Velocity</td>
</tr>
</tbody>
</table>

These two sorts of particles would therefore decay at 100%. CLEMENTS-PRICE (6) use a coefficient \( T < 1 \) dependent of the flow; we can estimate a new limit for the settling velocity \( q_e = q_e / T \). \( T \) is calculated with the help of the measure of velocity made by a Doppler laser anemometer (7) (3). The table 5 gives the limit for the settling velocity.

We might obtain again a 100% efficiency, as all the particles ending by going out, we obtain a 0% efficiency.

For the two previous models, we suppose there is no putting back into suspension of the deposited particles. So, in our case, this probability of putting back into suspension is equal to zero \( \delta = 0 \).

So: \( P_2 (t) = TTR = 1 - \alpha e^{-\lambda t} \) hence an efficiency of \( (1 - \alpha) \).

Plot theoretical and experimental curves \( 1 - \alpha = f(W/q_e) \) (see Figure 8).
Experimental results make a clear correlation, but which appreciably differs from the theoretical correlation. It seems that there is a limit value $W/q_e \approx 0.5$ for which the efficiency is equal to zero. This value could be interesting to be found experimentally, for it corresponds to the limit of the transported particles without sedimentation in the flow. There is also a value for $W/q_e$ for which all the particles will deposit ($W/q_e < 4$).
CONCLUSION

Let us recall first of all that for the used solid particles, we should have had a 100% theoretical efficiency, as all the particles end by getting out so an 0% efficiency.

This draining off of particles is effectuated according to two phases:
— one phase of fast drainage of the sediments which corresponds to the sedimentation of these ones. Let us note that the use of a CLEMENTS-PRICE model is possible to foresee the rate of deposited particles if dividing T by 4 or 5;
— a second phase of slow drainage which corresponds to the putting back into suspension of the sediments from the zones of sedimentations without it; it is impossible to find again the experimental results.

Let us note that we can have linked the phase of fast drainage to the structure of the mean flow and to the turbulence, and the phase of slow drainage to the passing of the big swirling structures in the zone of stable sedimentation of the flow (7).

Then, we have shown the influence of obstacles places in the sedimentation tank; an inlet plate favours the initial sedimentation and bars the nonputting back into suspension of this sedimentation; the combining of this two kinds of obstacles seems to optimize the working of the sedimentation tank under some conditions.

Finally, we have proposed r as a performance rate that tries to express by an only numerical value the working of the apparatus in these two phases. We think this rate ought to be retained to take into account the case of the real sedimentation tank and might permit after a practical study to link the scour frequency to the performance rate.

Let us not forget that we must replace this study in its "natural background", that is to say a continuous release of sediments into the sedimentation tank for variable flows that is being made in our Laboratory.
APPENDIX I - REFERENCES

3. ALQUIER, J.F. and DARTUS, D., Chaîne d'acquisition de mesures de répartition de vitesses à l'intérieur d'un décanteur rectangulaire, Rapport interne Nr. 81-1, IMFT, 1981.
15. PELLEREJ, M., Contribution à l'amélioration des performances d'un séparateur statique tourbillonnaire, Thèse de docteur-ingénieur, ENSEEIHT, IMFT, 1981.
APPENDIX II - NOTATION

\[ P_i(t) \quad \text{Probability to be in state } i \text{ at time } t \]
\[ \lambda \Delta t \quad \text{Probability for passing from one state to another} \]
\[ \mu \Delta t \]
\[ \gamma \Delta t \]
\[ \delta \Delta t \]
\[ \text{T.T.R.} \quad \text{Transit Time Repartition} \]
\[ 1 - \alpha \quad \text{Proportion of trapped particles} \]
\[ 1/\lambda_1 \quad \text{Average staying time in state 1 or 2} \]
\[ 1/\lambda_2 \]
\[ V_o \quad \text{Volume of the sedimentation tank} \]
\[ Q \quad \text{Flow} \]
\[ t_t \quad \text{Average staying time of liquid particles in the tank} \]
\[ t_s \quad \text{Average staying time of solid particles in the tank} \]
\[ r \quad \text{Performance rate} \]
\[ q_o \quad \text{Limit for the settling velocity with HAZEN's model} \]
\[ q_c \quad \text{Limit for the settling velocity with CLEMENTS PRICE's model} \]
\[ \bar{T} \quad \text{Coefficient of CLEMENTS PRICE} \]